



IIT ROORKEE



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CERTIFICATION COURSE

# QUANTITATIVE INVESTMENT MANAGEMENT

## LECTURE 47 Brownian Motion

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**Hence, our complete set of assumptions is :**

**(i)  $W_n(0) = 0$**

**(ii) no of steps =  $n$  so that, layer spacing  $T/n$ ,**

**(iii) up and down jumps equal and of size  $\sqrt{\frac{T}{n}}$ ,**

**(iv) up and down probabilities everywhere equal to  $\frac{1}{2}$ .**



**Mathematically, our random walk consists of  $n$  Independent identically distributed random variables  $Y_i$**

$$Y_i = \sqrt{\frac{T}{n}} X_i \text{ where } X_i \text{ are IIDs defined by}$$

$$X_i = \begin{cases} +1 & \text{with } p(X_i = +1) = 1/2 \\ -1 & \text{with } p(X_i = -1) = 1/2 \end{cases}$$

**and the recursive relation**

$$\begin{aligned} W_n^Y(T) &= W_n^Y\left(n \cdot \frac{T}{n}\right) = W_n^Y\left((n-1) \cdot \frac{T}{n}\right) + Y_n \\ &= W_n^Y\left((n-2) \cdot \frac{T}{n}\right) + Y_{n-1} + Y_n = \sum_{i=1}^n Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^n X_i \end{aligned}$$

$$\mathbf{E} \left[ \mathbf{W}_n^Y(\mathbf{T}) \right] = \mathbf{E} \left[ \sum_{i=1}^n \mathbf{Y}_i \right] = \left[ \sum_{i=1}^n \mathbf{E}(\mathbf{Y}_i) \right] = \mathbf{0}$$

$$\mathbf{E} \left[ \mathbf{W}_n^Y(\mathbf{T}) \right]^2 = \mathbf{E} \left[ \sum_{i=1}^n \mathbf{Y}_i \right]^2 = \mathbf{E} \left[ \sum_{i=1}^n \mathbf{Y}_i^2 \right] = \left[ \sum_{i=1}^n \mathbf{E}(\mathbf{Y}_i^2) \right]$$

$$\left[ \sum_{i=1}^n \mathbf{E} \left( \frac{\mathbf{T}}{\mathbf{n}} \mathbf{X}_i^2 \right) \right] = \frac{\mathbf{T}}{\mathbf{n}} \left[ \sum_{i=1}^n \mathbf{E}(\mathbf{X}_i^2) \right] = \frac{\mathbf{T}}{\mathbf{n}} \mathbf{n} = \mathbf{T}. \text{ Hence } \sigma_T^2 = \mathbf{T}$$

Also  $\mathbf{E}(\mathbf{Y}_i) = \mathbf{0}$  and so  $\sigma_i^2 = \mathbf{E}(\mathbf{Y}_i^2) = \frac{\mathbf{T}}{\mathbf{n}} [\mathbf{E}(\mathbf{X}_i^2)] = \frac{\mathbf{T}}{\mathbf{n}}$

**Clearly,  $E[W_n^Y(T)] = 0$ ;  $\text{Var}[W_n^Y(T)] = T$**   
**Please note so far no limits have been taken.**

**For an arbitrary  $t$  in  $(0, T)$ , no of steps =  $\frac{nt}{T}$ . Hence,**

**In analogy with  $W_n^Y(T) = W_n^Y\left(n \cdot \frac{T}{n}\right) = \sum_{i=1}^n Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^n X_i$**

**we have  $W_n^Y(t) = W_n^Y\left(\frac{nt}{T} \cdot \frac{T}{n}\right) = \sum_{i=1}^{nt/T} Y_i = \sqrt{\frac{T}{n}} \sum_{i=1}^{nt/T} X_i$**

**$E[W_n^Y(t)] = 0$ ;  $E[W_n^Y(t)]^2 = \frac{T}{n} \cdot \frac{nt}{T} = t$ ;  $\text{Var}[W_n^Y(t)] = t$**

- To move from the above discrete version of random walk to the continuous time version (formally called Brownian motion) several approaches can be adopted e.g.
- The Central Limit Theorem;
- The Diffusion PDE;
- The Stirling Approximation.

# CENTRAL LIMIT THEOREM

- Let  $Y_i; i=1,2,\dots,n$  be independent identically distributed random variables **each** with finite mean and variance  $\mu$  and  $\sigma^2$  respectively. Then the following expression is distributed as a standard normal variate.

$$Z_{\infty} = \lim_{n \rightarrow \infty} \frac{\left( \sum_{i=1}^n Y_i \right) - n\mu}{\sqrt{n\sigma^2}}$$

- The cardinal feature of the Central Limit Theorem is that the limiting distribution invariably approaches the normal distribution **irrespective of the underlying distributions** of the random variables themselves.
- **The only requirement is the finiteness of the means and variances of these underlying distributions.**

# BM AS A LIMITING CASE OF SCALED RANDOM WALK

- In the limit that the number of time steps approaches infinity, the aforesaid construction of a scaled random walk converges to a mathematical structure called Brownian motion that has certain well defined mathematical properties and plays a vital role in the modeling of stochastic processes.
- BM is also sometimes called a Wiener Process



Mathematically, our  $Y$  random walk consists of a sequence of Independent identically distributed random variables  $Y_i$

$$W_n^Y(t) = \sum_{i=1}^{nt/T} Y_i; \quad Y_i = \sqrt{\frac{T}{n}} X_i \text{ where } X_i \text{ are IIDs defined by}$$

$$X_i = \begin{cases} +1 & \text{with } p(X_i = +1) = 1/2 \\ -1 & \text{with } p(X_i = -1) = 1/2 \end{cases}$$

Obviously,  $E(Y_i) = 0; E(Y_i)^2 = \frac{T}{n}$  so that  $\sigma_i^2 = \frac{T}{n}$

**In the case of the Y random walk**

$$\mu_i = E(Y_i) = 0; E(Y_i^2) = \frac{T}{n}, \sigma_i^2 = \frac{T}{n} \quad \forall i$$

Hence, for arbitrary  $t \in (0, T)$  no of steps =  $\frac{nt}{T}$

For modeling each step we need one IIDrv  $Y_i$

Hence, no of IIDs  $Y_i = \frac{nt}{T}$ . By CLT  $\lim_{\frac{nt}{T} \rightarrow \infty} \frac{\sum_{i=1}^{nt/T} Y_i - \frac{nt}{T} \mu}{\sqrt{\frac{nt}{T} \sigma_i^2}}$

$$= \lim_{nt/T \rightarrow \infty} \frac{\sum_{i=1}^{nt/T} Y_i}{\sqrt{\frac{nt}{T} \frac{T}{n}}} = \lim_{nt/T \rightarrow \infty} \frac{\sum_{i=1}^{nt/T} Y_i}{\sqrt{t}} \xrightarrow{\text{distribution}} N(0, 1)$$

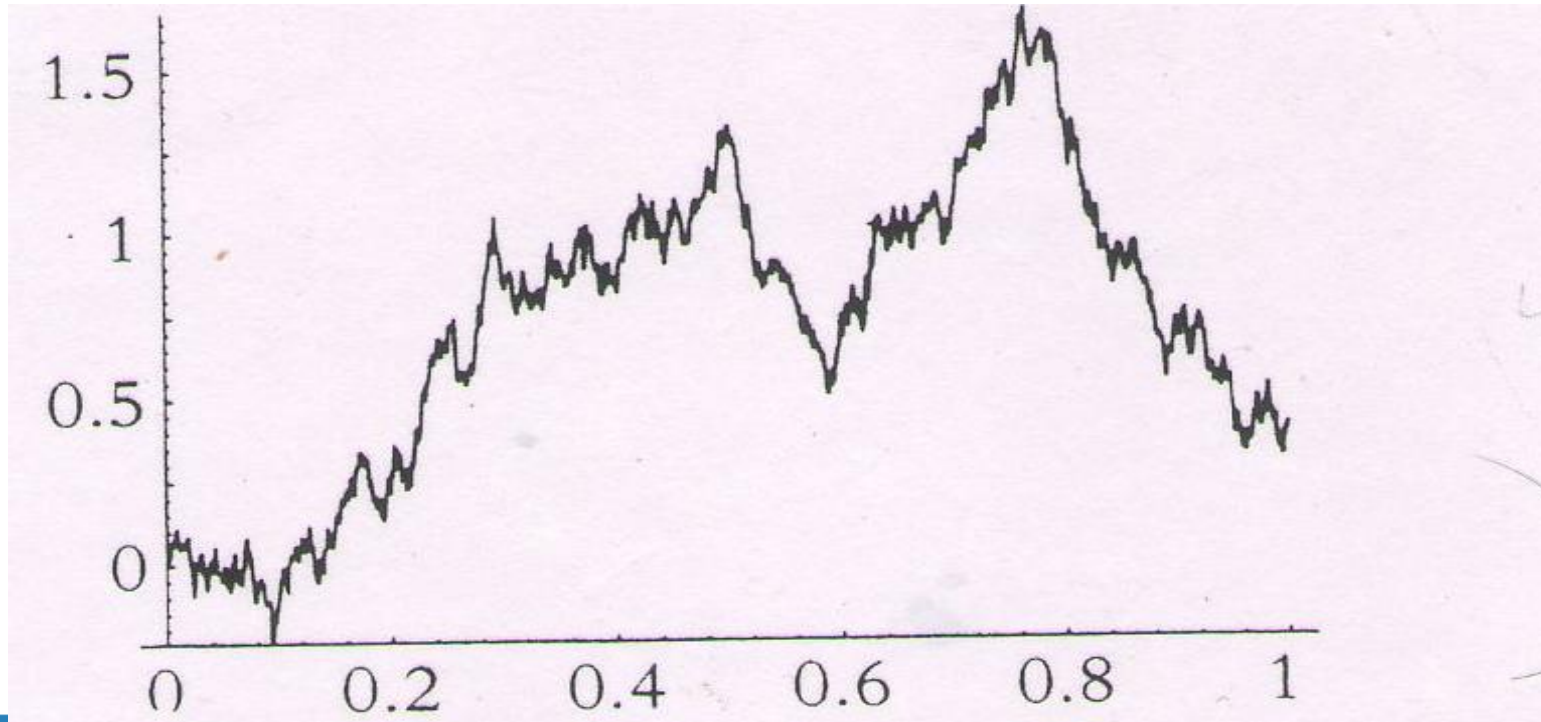
From above  $\lim_{nt/T \rightarrow \infty} \frac{\sum_{i=1}^{nt/T} Y_i}{\sqrt{t}} \xrightarrow{\text{distribution}} N(0, 1)$

Or  $\lim_{nt/T \rightarrow \infty} \sum_{i=1}^{nt/T} Y_i \xrightarrow{\text{distribution}} N(0, t)$  so that

$$W^Y(t) = W_{\infty}^Y(t) = W_{n \rightarrow \infty}^Y(t) = W_{nt/T \rightarrow \infty}^Y(t)$$

$$= \lim_{nt/T \rightarrow \infty} \sum_{i=1}^{nt/T} Y_i \xrightarrow{\text{distribution}} N(0, t)$$

# BROWNIAN MOTION



The process  $W = (W(t) : t \geq 0)$  is a

Brownian motion if and only if

(i) CONTINUITY :  $W(t)$  is continuous, and  $W(0) = 0$ ,

(ii) DISTRIBUTION OF  $W(t)$  : The value of  $W(t)$  is distributed as a normal random variable  $N(0, t)$ ,

(iii) DISTRIBUTION OF INCREMENTS : The increment  $W(s + t) - W(s)$  is distributed as a normal  $N(0, t)$ , and is independent of the history of what the process did up to times.



#### **(iv) REPRESENTATION IN TERMS OF $z$**

**we can express an increment of BM as  $dW(t) = \sqrt{dt}Z$**

**where  $Z$  is  $N(0,1)$  distributed normal variate.**

#### **(v) DIFFERENTIABILITY**

**The process  $W(t)$  is not differentiable at any point  $t$**

#### **(vi) FRACTALITY**

**BM is a self replicating object i.e. a FRACTAL.**

# BROWNIAN MOTION WITH DRIFT OR GENERALIZED WIENER PROCESS

- The infinitesimal increment of a generalized Wiener process can be expressed as:  $dx = \mu dt + \sigma dW$
- Two components:
  - (1) a non-stochastic component given by  $\mu dt$  representing the drift and
  - (2) a stochastic component given by  $\sigma dW$  where  $dW = z\sqrt{dt}$ ,  $z$  is  $N(0,1)$  and  $\sigma$  is a scaling factor of the dispersion of the process.

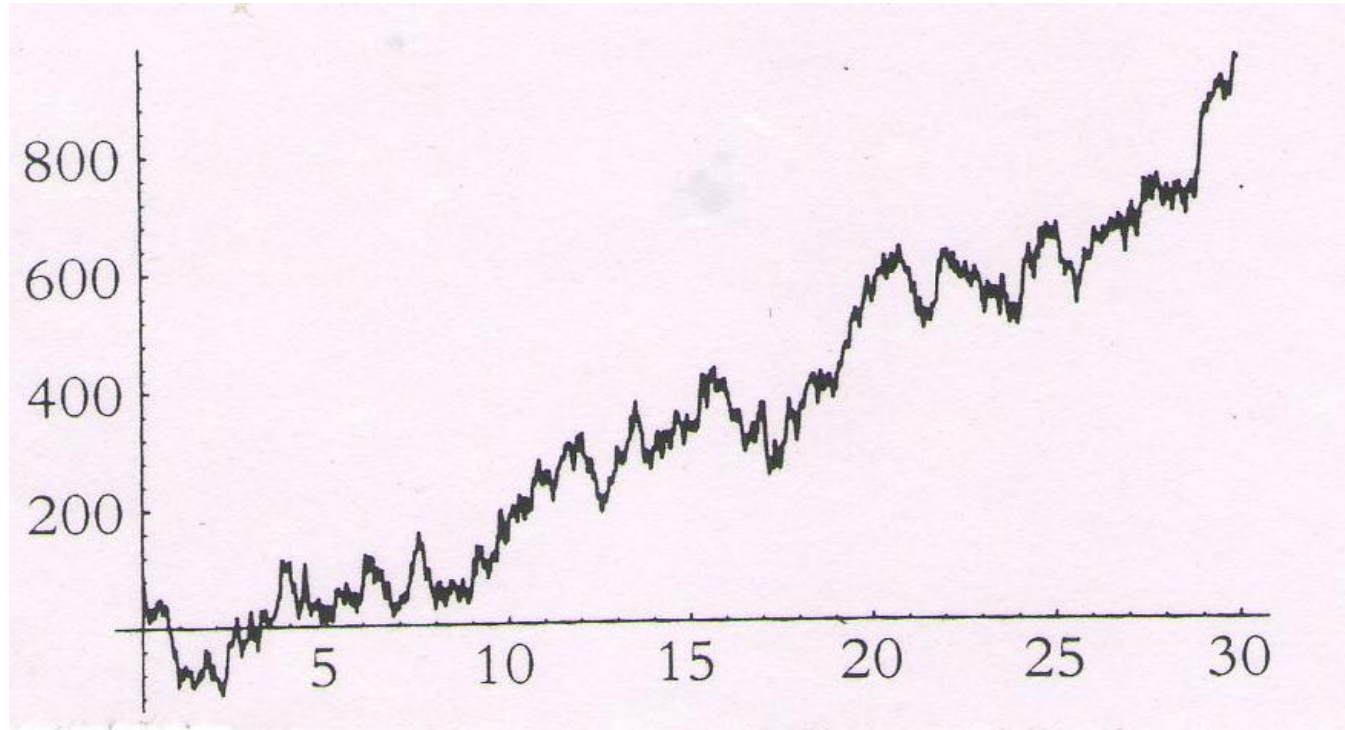


# MEAN AND VARIANCE OF GENERALIZED WIENER PROCESS

- For the generalized Wiener process
  - $dx = \mu dt + \sigma dW$
  - Mean =  $\mu dt$
  - Variance =  $\sigma^2 dt$



# BROWNIAN MOTION WITH DRIFT



| Random Walks                   | Time Length /Layer Spacing | Jump Size                           | Exp Value | Variance     |
|--------------------------------|----------------------------|-------------------------------------|-----------|--------------|
| Single Step RW                 | T                          | $\pm 1$                             | 0         | 1            |
| Two Step RW                    | T/2                        | $\pm 1$                             | 0         | 2            |
| n-Step RW                      | T/n                        | $\pm 1$                             | 0         | n            |
| n-Step Scaled RW               | T/n                        | $\pm \sqrt{(T/n)}$                  | 0         | T            |
| n-Step Scaled RW               | T/n                        | $\pm \sigma \sqrt{(T/n)}$           | 0         | $\sigma^2 T$ |
| n-Step Scaled RW<br>With Drift | T/n                        | $(\mu T/n) \pm \sigma \sqrt{(T/n)}$ | $\mu T$   | $\sigma^2 T$ |
| Brownian Motion                | ->0                        | ->0                                 | 0         | T            |
| Scaled BM                      | ->0                        | ->0                                 | 0         | $\sigma^2 T$ |

# DRIFT RATE & VARIANCE RATE

- The mean change per unit time for a stochastic process is called the drift rate.
- The variance per unit time is called the variance rate.

